# On the survival of a class of subcritical branching processes in random environment

Vincent Bansaye, Vladimir Vatutin

# On the survival of a class of subcritical branching processes in random environment

Vincent Bansaye* and Vladimir Vatutin†

December 19, 2013

**Abstract**

Let $Z_n$ be the number of individuals in a subcritical BPRE evolving in the environment generated by iid probability distributions. Let $X$ be the logarithm of the expected offspring size per individual given the environment. Assuming that the density of $X$ has the form

$$p_X(x) = x^{-\beta-1} l_0(x) e^{-\rho x}$$

for some $\beta > 2$, a slowly varying function $l_0(x)$ and $\rho \in (0,1)$, we find the asymptotic of the survival probability $\mathbb{P}(Z_n > 0)$ as $n \to \infty$, prove a Yaglom type conditional limit theorem for the process and describe the conditioned environment. The survival probability decreases exponentially with an additional polynomial term related to the tail of $X$. The proof uses in particular a fine study of a random walk (with negative drift and heavy tails) conditioned to stay positive until time $n$ and to have a small positive value at time $n$, with $n \to \infty$.

## 1 Introduction

We consider the model of branching processes in random environment introduced by Smith and Wilkinson [16]. The formal definition of these processes looks as follows. Let $\mathfrak{N}$ be the space of probability measures on $\mathbb{N}_0 = \{0, 1, 2, ...\}$. Equipped with the metric of total variation $\mathfrak{N}$ becomes a Polish space. Let $\mathfrak{e}$ be a random variable taking values in $\mathfrak{N}$. An infinite sequence $\mathcal{E} = (\mathfrak{e}_1, \mathfrak{e}_2, \ldots)$ of i.i.d. copies of $\mathfrak{e}$ is said to form a *random environment*. A sequence of $\mathbb{N}_0$-valued random variables $Z_0, Z_1, \ldots$ is called a *branching process in the random environment* $\mathcal{E}$, if $Z_0$ is independent of $\mathcal{E}$ and, given $\mathcal{E}$, the process $Z = (Z_0, Z_1, \ldots)$ is a Markov chain with

$$\mathcal{L}\left(Z_n \mid Z_{n-1} = z_{n-1},\, \mathcal{E} = (e_1, e_2, \ldots)\right) \;=\; \mathcal{L}\left(\xi_{n1} + \cdots + \xi_{nz_{n-1}}\right) \qquad (1)$$

*CMAP, Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex, France; e-mail: bansaye@polytechnique.edu

†Department of Discrete Mathematics, Steklov Mathematical Institute, 8, Gubkin str., 119991, Moscow, Russia; e-mail: vatutin@mi.ras.ru

for every $n \geq 1$, $z_{n-1} \in \mathbb{N}_0$ and $e_1, e_2, \ldots \in \mathfrak{N}$, where $\xi_{n1}, \xi_{n2}, \ldots$ are i.i.d. random variables with distribution $\mathfrak{e}_n$. Thus,

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_{ni} \tag{2}$$

and, given the environment, $Z$ is an ordinary inhomogeneous Galton-Watson process. We will denote the corresponding probability measure and expectation on the underlying probability space by $\mathbb{P}$ and $\mathbb{E}$, respectively.

Let

$$X = \log\left(\sum_{k\geq 0} k\mathfrak{e}\left(\{k\}\right)\right), \qquad X_n = \log\left(\sum_{k\geq 0} k\mathfrak{e}_n\left(\{k\}\right)\right),\ n = 1, 2, ...,$$

be the logarithms of the expected offspring size per individual in the environments and

$$S_0 = 0,\ S_n = X_1 + \cdots + X_n, n \geq 1,$$

be their partial sums.

This paper deals with the subcritical branching processes in random environment, i.e., in the sequel we always assume that

$$\mathbb{E}\left[X\right] = -b < 0. \tag{3}$$

The subcritical branching processes in random environment admit an additional classification, which is based on the properties of the moment generating function

$$\varphi(t) = \mathbb{E}\left[e^{tX}\right] = \mathbb{E}\left[\left(\sum_{k\geq 0} k\mathfrak{e}\left(\{k\}\right)\right)^t\right], \quad t \geq 0.$$

Clearly, $\varphi'(0) = \mathbb{E}\left[X\right]$. Let

$$\rho_+ = \sup\left\{t \geq 0 : \varphi(t) < \infty\right\}$$

and $\rho_{min}$ be the point where $\varphi(t)$ attains its minimal value on the interval $[0, \rho_+ \wedge 1]$. Then a subcritical branching process in random environment is called

weakly subcritical if $\rho_{min} \in (0, \rho_+ \wedge 1)$,
intermediately subcritical if $\rho_{min} = \rho_+ \wedge 1 > 0$ and $\varphi'(\rho_{min}) = 0$,
strongly subcritical if $\rho_{min} = \rho_+ \wedge 1$ and $\varphi'(\rho_{min}) < 0$.

Note that this classification is slightly different from that given in [9]. Weakly subcritical and intermediately subcritical branching processes have been studied in [14, 1, 2, 3] in detail. Let us recall that $\varphi'(\rho_+ \wedge 1) > 0$ for the weakly subcritical case.

The strongly subcritical case is also well studied for the case $\rho_+ \geq 1$, i.e., if $\rho_{min} = \rho_+ \wedge 1 = 1$ and $\varphi'(1) < 0$. In particular, it was shown in [14] and refined in [5] that if $\varphi'(1) = \mathbb{E}\left[Xe^X\right] < 0$ and $\mathbb{E}\left[Z_1 \log^+ Z_1\right] < \infty$ then, as $n \to \infty$

$$\mathbb{P}\left(Z_n > 0\right) \sim K\left(\mathbb{E}\left[\xi\right]\right)^n, \; K > 0, \tag{4}$$

and, in addition,

$$\lim_{n\to\infty} \mathbb{E}\left[s^{Z_n} | Z_n > 0\right] = \Psi(s), \tag{5}$$

where $\Psi(s)$ is the probability generating function of a proper nondegenerate random variable on $\mathbb{Z}_+$. This statement is actually an extension of the classical result for the ordinary subcritical Galton-Watson branching processes.

## 2 Main results

Our main concern in this paper is the strongly subcritical branching processes in random environment with $\rho_+ \in (0, 1)$. More precisely, we assume that the following condition is valid:

**Hypothesis A.** The distribution of $X$ has density

$$p_X(x) = \frac{l_0(x)}{x^{\beta+1}} e^{-\rho x}, \tag{6}$$

where $l_0(x)$ is a function slowly varying at infinity, $\beta > 2$, $\rho \in (0, 1)$ and, in addition,

$$\varphi'(\rho) = \mathbb{E}\left[Xe^{\rho X}\right] < 0. \tag{7}$$

This assumption can be relaxed by assuming that $p_X(x)$ is the density of $X$ for $x$ large enough, or that the tail distribution $\mathbb{P}(X \in [x, x+\Delta)) \sim \int_x^{x+\Delta} p_X(y)dy$ for $x \to \infty$ (uniformly with respect to $\Delta \leq 1$).
Clearly, $\rho = \rho_+ < 1$ under Hypothesis A. Observe that the case $\rho = \rho_+ = 0$ not included in Hypothesis A has been studied in [17] and yields a new type of the asymptotic behavior of subcritical branching processes in random environment. Namely, it was established that, as $n \to \infty$

$$\mathbb{P}\left(Z_n > 0\right) \sim K\mathbb{P}\left(X > nb\right) = K\frac{l_0(nb)}{(nb)^\beta}, \; K > 0, \tag{8}$$

so that the survival probability decays with a polynomial rate only. Moreover, for any $\varepsilon > 0$, some constant $\sigma > 0$ and any $x \in \mathbb{R}$

$$\mathbb{P}\left(\frac{\log Z_n - \log Z_{[n\varepsilon]} + n(1-\varepsilon)b}{\sigma\sqrt{n}} \leq x \,|\, Z_n > 0\right) = \mathbb{P}\left(B_1 - B_\varepsilon \leq x\right)$$

where $B_t$ is a standard Brownian motion. Therefore, given the survival of the population up to time $n$, the number of individuals in the process at this moment tends to infinity as $n \to \infty$ that is not the case for other types of subcritical processes in random environment.

The goal of the paper is to investigate the asymptotic behavior of the survival probability of the process meeting Hypothesis A and to prove a Yaglom-type conditional limit theorem for the distribution of the number of individuals. To this aim we additionally assume that the sequence of conditional probability measures

$$\mathbb{P}^{[x]}\left(\cdot\right)=\mathbb{P}\left(\cdot\mid X=x\right)$$

is well defined for $x\to\infty$ under Hypothesis $A$. We provide in Section 3 natural examples when this assumption and Hypothesis $B$ below are valid.

Denote by $\mathfrak{L}=\{\mathcal{L}\}$ the set of all proper probability measures $\mathcal{L}(\cdot)$ of non-negative random variables. Our next condition concerns the behavior of the measures $\mathbb{P}^{[x]}$ as $x\to\infty$ :

**Hypothesis B.** There exists a probability measure $\mathbb{P}^{*}$ on $\mathfrak{L}$ such that, as $x\to\infty$,

$$\mathbb{P}^{[x]}\Longrightarrow\mathbb{P}^{*}$$

where the symbol $\Longrightarrow$ stands for the weak convergence of measures.

Setting

$$a=-\frac{\varphi^{\prime}\left(\rho\right)}{\varphi\left(\rho\right)}>0,$$

we are now ready to formulate the first main result of the paper.

**Theorem 1** *If*

$$\mathbb{E}\left[-\log\left(1-\mathfrak{e}\left(\{0\}\right)\right)\right]<\infty,\quad\mathbb{E}\left[e^{-X}\sum_{k\geq1}\mathfrak{e}\left(\{k\}\right)k\log k\right]<\infty\tag{9}$$

*and Hypotheses A and B are valid, then there exists a constant $C_0>0$ such that, as $n\to\infty$*

$$\mathbb{P}\left(Z_{n}>0\right)\sim C_{0}\rho\varphi^{n-1}\left(\rho\right)e^{an\rho}\mathbb{P}\left(X>an\right)\sim C_{0}\rho\varphi^{n-1}\left(\rho\right)\frac{l_{0}(n)}{\left(an\right)^{\beta+1}}.\tag{10}$$

We stress that $\varphi(\rho)\in(0,1)$. Moreover, the explicit form of $C_0$ can be found in (49). The proof is given in Section 6. We now quickly explain this asymptotic behavior and give at the same time an idea of the proof. In the next Section, some examples of processes satisfying the assumptions required in Theorem 1 can be found.

For the proof, we introduce in Section 4.1 a new probability measure $\mathbf{P}$. Under this new probability measure, the random walk $\mathbf{S}=(S_n,n\geq0)$ has the drift $-a<0$ and the heavy tail distribution of its increments has polynomial decay $\beta$. Adding that $\mathbf{E}\left[\exp(\rho X)\right]=\varphi\left(\rho\right)$, we will get the survival probability as

$$\varphi^{n}\left(\rho\right)\mathbf{E}\left[e^{-\rho S_{n}}\mathbf{P}(Z_{n}>0|\mathfrak{e})\right]\approx const\times\varphi^{n}\left(\rho\right)\mathbf{P}(L_{n}\geq0,S_{n}\leq N)$$

where $L_n$ is the minimum of the random walk up to time $n$ and $N$ is (large but) fixed.

We then make use of the properties of random walks with negative drift and heavy tails of increments established in [7] to show that

$$\mathbf{P}(L_n \geq 0, S_n \leq N) \approx const \times \mathbf{P}(X_1 \in [an - M\sqrt{n}, an + M\sqrt{n}], S_n \in [0,1])$$

for $n$ large enough and conclude using the central limit theorem.

Our second main result is a Yaglom-type conditional limit theorem.

**Theorem 2** *Under the conditions of Theorem 1,*

$$\lim_{n\to\infty} \mathbb{E}\left[s^{Z_n} | Z_n > 0\right] = \Omega(s),$$

*where* $\Omega(s)$ *is the probability generating function of a proper nondegenerate random variable on* $\mathbb{Z}_+$.

We see that, contrary to the case $\rho_{min} = \rho_+ \wedge 1 = 0$ this Yaglom-type limit theorem has the same form as for the ordinary Galton-Watson subcritical processes.

Introduce a sequence of generating functions

$$f_n(s) = f(s; \mathfrak{e}_n) = \sum_{k=0}^{\infty} \mathfrak{e}_n(\{k\}) s^k, \qquad 0 \leq s \leq 1,$$

specified by the environmental sequence $(\mathfrak{e}_1, \mathfrak{e}_2, ..., \mathfrak{e}_n, ...)$ and denote

$$f_{j,n} = f_{j+1} \circ \cdots \circ f_n, \quad f_{n,j} = f_n \circ \cdots \circ f_{j+1} \quad (j < n), \; f_{n,n} = Id. \tag{11}$$

For every pair $n \geq j \geq 1$, we define a random function $g_j : \mathbb{R}_+ \to [0,1]$, a tuple of random variables

$$W_{n,j} = \frac{1 - f_{n,j}(0)}{e^{S_n - S_j}} \tag{12}$$

and a random variable $W_j$ on $[0,1]$ such that

(i) the distribution of $g_j$ is given by $\mathbb{P}^*$ and that of $W_j$ is given by the (common) distribution of $\lim_{n\to\infty} W_{n,j}$, which exists by monotonicity;

(ii) $f_{0,j-1}$, $g_j$ and $(W_{n,j}, W_j, f_k : k \geq j+1)$ are independent for each $n \geq j$ (it is always possible, the initial probability space being extended if required).

Then we can set

$$c_j = \int_{-\infty}^{\infty} \mathbb{E}\left[1 - f_{0,j-1}(g_j(e^v W_j))\right] e^{-\rho v} dv$$

and state the following result. It describes the environments that provide survival of the population until time $n$.

**Theorem 3** *For each $j \geq 1$,*
*i) the following limit exists*

$$\pi_j = \lim_{n\to\infty} \mathbb{P}(X_j \geq an/2|Z_n > 0) = \frac{c_j \varphi^{-j}(\rho)}{\sum_{k\geq 1} c_k \varphi^{-k}(\rho)}.$$

*ii) for each measurable and bounded function $F : \mathbb{R}^j \to \mathbb{R}$ and each family of measurable uniformly bounded functions $F_n : \mathbb{R}^{n+1} \to \mathbb{R}$ the difference*

$$\begin{aligned} &\mathbb{E}\left[F(S_0,\ldots,S_{j-1})F_{n-j}(S_n - S_{j-1}, X_{j+1},\ldots,X_n)|Z_n > 0,\ X_j \geq an/2\right] \\ &\quad - c_j^{-1}\mathbb{E}\left[F(S_0,\ldots,S_{j-1})\int_{-\infty}^{\infty} F_{n-j}(v, X_n, ..., X_{j+1})G_{j,n}(v)dv\right] \end{aligned}$$

*goes to* 0 *as $n \to \infty$, where*

$$G_{j,n}(v) := (1 - f_{0,j-1}(g_j(e^v W_{n,j})))\, e^{-\rho v}.$$

Detailed descriptions of the properties of the random function $g_j$ and the random variable $W$ are given by (21) and before the proof of Lemma 16, respectively. We refer to [2, 3, 4, 5] for similar questions in the subcritical and critical regimes. Here the conditioned environment is different since a big jump appear, whereas the rest of the random walk looks like the original one. Let us now focus on this exceptional environment explaining the survival event and give a more explicit result.

**Corollary 4** *Let $\varkappa = \inf\{j \geq 1 : X_j \geq an/2\}$. Under $\mathbb{P}$, conditionally on $Z_n > 0$, $\varkappa$ converges in distribution to a proper random variable whose distribution is given by $(\pi_j : j \geq 1)$. Moreover, conditionally on $\{Z_n > 0, X_j \geq an/2\}$, the distribution law of $(X_\varkappa - an)/(VarX\sqrt{n})$ converges to a law $\mu$ specified by*

$$\mu(B) = c_j^{-1}\mathbb{E}\left[1(G \in B)\int_{-\infty}^{\infty} (1 - f_{0,j-1}(g_j(e^v W_j)))\, e^{-\rho v} dv\right]$$

*for any Borel set $B \subset \mathbb{R}$, where $G$ is a centered gaussian random variable with variance $VarX$, which is independent of $(f_{0,j-1}, g_j)$.*

# 3 Examples

We provide here some examples meeting the conditions of Theorem 1. Thus, we assume that Hypothesis $A$ is valid and we focus on the existence and convergence of $\mathbb{P}^{[x]}$. Let us first deal with the existence of random reproduction laws $\mathfrak{e}$ for which the conditional probability

$$\mathbb{P}^{[x]}(\cdot) = \mathbb{P}(\cdot \mid X = x)$$

is well defined.

*Example 0.* Assume that the environment $\mathfrak{e}$ takes its values in some set $\mathcal{M}$ of probability measures such that for all $\mu, \nu \in \mathcal{M}$

$$\sum_{k\geq 0} k\mu(k) < \sum_{k\geq 0} k\nu(k) \Rightarrow \mu \leq \nu,$$

where $\mu \leq \nu$ means that $\forall l \in \mathbb{N}, \mu[l,\infty) \leq \nu[l,\infty)$. We note that Hypothesis $A$ ensures that $\mathbb{P}(\cdot|X \in [x, x+\epsilon))$ is well defined. Then, for every $H : \mathcal{M} \to \mathbb{R}^+$ which is non decreasing in the sense that $\mu \leq \nu$ implies $H(\mu) \leq H(\nu)$, we get that the functional

$$\mathbb{E}\left[H(\mathfrak{e})|X \in [x, x+\epsilon)\right]$$

decreases to some limit $p(H)$ as $\epsilon \to 0$. Thus, writing $H_{l,y}(\mu) = 1$ if $\mu[l,\infty) \geq y$ and 0 otherwise, we can define $\mathbb{P}^{[x]}$ via

$$\mathbb{P}^{[x]}\left(\mathfrak{e}[l,\infty) \geq y\right) = p(H_{l,y})$$

to get the expected conditional probability.

Let us now focus on Hypothesis B.

*Example 1.* Let $f(s;\mathfrak{e}) = \sum_{k\geq 0} \mathfrak{e}\left(\{k\}\right) s^k$ be the (random) probability generating function corresponding to the random measure $\mathfrak{e} \in \mathfrak{N}$ and let (with a slight abuse of notation) $\xi = \xi\left(\mathfrak{e}\right) \geq 0$ be the integer-valued random variable with probability generating function $f(s;\mathfrak{e})$, i.e., $f(s;\mathfrak{e}) = E\left[s^{\xi(\mathfrak{e})}\right]$.

It is not difficult to understand that if $\mathbb{E}\left[\log f'(1;\mathfrak{e})\right] < 0$ and there exists a deterministic function $g(\lambda), \lambda \geq 0$, with $g(\lambda) < 1, \lambda > 0$, and $g(0) = 1$, such that, for every $\varepsilon > 0$

$$\lim_{y\to\infty} \mathbb{P}\left(\mathfrak{e} : \sup_{0\leq\lambda<\infty} \left| f\left(e^{-\lambda/y};\mathfrak{e}\right) - g(\lambda)\right| > \varepsilon \,\middle|\, f'(1;\mathfrak{e}) = y = e^x\right) = 0,$$

then Hypothesis B is satisfied for the respective subcritical branching process.

We now give two more explicit examples for which Hypothesis B holds true and note that mixing the two classes described in these examples would provide a more general family which satisfies Hypothesis B.

Let $\mathfrak{N}_f \subset \mathfrak{N}$ be the set of probability measures on $\mathbb{N}_0$ such that

$$e = e\left(t, y\right) \in \mathfrak{N}_f \iff f\left(s; e\right) = 1 - t + \frac{t}{1 + yt^{-1}\left(1-s\right)}$$

where $t \in (0, 1]$ and $y \in (0, \infty)$, and let $\mathfrak{L}_g \subset \mathfrak{L}$ be the set of probability measures such that

$$L = L\left(t, y\right) \in \mathfrak{L}_g \iff g(t,\lambda) = \int e^{-\lambda y} L(t, dy) = 1 - t + \frac{t^2}{t+\lambda}.$$

Let, further, $\mathcal{B}=\mathcal{B}_1\times\mathcal{B}_2\subset(0,1]\times(0,\infty)$ be a Borel set. We write

$$e=e\left(t,y\right)\in T\left(\mathcal{B}\right)\subseteq\mathfrak{N}_f \text{ if } \left(t,y\right)\in\mathcal{B}$$

and

$$L=L(t,y)\in T\left(\mathcal{B}_1\right)\subseteq\mathfrak{L}_g \text{ if } t\in\mathcal{B}_1.$$

Let $(\theta,\zeta)$ be a pair of random variables with values in $(0,1]\times(0,\infty)$ such that for a measure $P^*\left(\cdot\right)$ with support on $(0,1]$ and any Borel set $\mathcal{B}_1\subseteq(0,1]$,

$$\lim_{x\to\infty}P\left(\theta\in\mathcal{B}_1|\zeta=x\right)=P^*\left(\theta\in\mathcal{B}_1\right)$$

exists.

With this notation in view we describe the desired two examples.
*Example 2.* Assume that the support of the probability measure $\mathbb{P}$ is concentrated on the set $\mathfrak{N}_f$ only and the random environment $\mathfrak{e}$ is specified by the relation

$$\mathfrak{e}=e\left(\theta,\zeta\right)\iff f\left(s;\mathfrak{e}\right)=1-\theta+\frac{\theta^2}{\theta+\zeta\left(1-s\right)}.$$

Clearly, $\log f'\left(1;\mathfrak{e}\right)=\log\zeta$. Thus,

$$\mathbb{P}\left(e\left(\theta,\zeta\right)\in T\left(\mathcal{B}\right)\right)=\mathbb{P}\left(f\left(s;\mathfrak{e}\right):\left(\theta,\zeta\right)\in\mathcal{B}\right)$$

and if $\mathcal{B}=\mathcal{B}_1\times\{x\}$ then

$$\begin{aligned}\lim_{x\to\infty}\mathbb{P}\left(f\left(e^{-\lambda\zeta^{-1}};\mathfrak{e}\right):\left(\theta,\zeta\right)\in\mathcal{B}|\zeta=e^x\right)&=P^*\left(\theta\in\mathcal{B}_1\right)\\&=\mathbb{P}^*\left(g(\lambda;\theta):\theta\in\mathcal{B}_1\right)=\mathbb{P}^*\left(L(\theta;y)\in T\left(\mathcal{B}_1\right)\right).\end{aligned}$$

Note that if $P\left(\theta=1|\zeta=x\right)=1$ for all sufficiently large $x$ we get a particular case of Example 1.

*Example 3.* If the support of the environment is concentrated on probability measures $\mathfrak{e}\in\mathfrak{N}$ such that, for any $\varepsilon>0$

$$\lim_{y\to\infty}\mathbb{P}\left(\mathfrak{e}:\left|\frac{\xi(\mathfrak{e})}{f'(1;\mathfrak{e})}-1\right|>\varepsilon\;\Big|\;f'(1;\mathfrak{e})=e^X=y\right)=0\tag{13}$$

and the density of the random variable $X=\log f'(1;\mathfrak{e})$ is positive for all sufficiently large $x$, then $g(\lambda)=e^{-\lambda}$. Condition (13) is satisfied if, for instance,

$$\lim_{y\to\infty}\mathbb{P}\left(\mathfrak{e}:\frac{Var\xi(\mathfrak{e})}{\left(f'(1;\mathfrak{e})\right)^2}>\varepsilon\;\Big|\;f'(1;\mathfrak{e})=y\right)=0.$$

# 4 Preliminaries

## 4.1 Change of probability measure

A nowadays classical technique of studying subcritical branching processes in random environment (see, for instance, [14, 4, 2, 3] ) is similar to that one used to investigate standard random walks satisfying the Cramer condition. Namely, denote by $\mathcal{F}_n$ the $\sigma-$algebra generated by the tuple $(\mathfrak{e}_1, \mathfrak{e}_2, ..., \mathfrak{e}_n; Z_0, Z_1, ..., Z_n)$ and let $\mathbb{P}^{(n)}$ be the restriction of $\mathbb{P}$ to $\mathcal{F}_n$. Setting

$$m = \varphi(\rho) = \mathbb{E}\left[e^{\rho X}\right],$$

we introduce another probability measure $\mathbf{P}$ by the following change of measure

$$d\mathbf{P}^{(n)} = m^{-n} e^{\rho S_n} d\mathbb{P}^{(n)}, \; n = 1, 2, ... \tag{14}$$

or, what is the same, for any random variable $Y_n$ measurable with respect to $\mathcal{F}_n$ we let

$$\mathbf{E}\left[Y_n\right] = m^{-n}\mathbb{E}\left[Y_n e^{\rho S_n}\right]. \tag{15}$$

By (7),

$$\mathbf{E}\left[X\right] = m^{-1}\mathbb{E}\left[X e^{\rho X}\right] = \varphi'(\rho)/\varphi(\rho) = -a < 0. \tag{16}$$

Applying a Tauberian theorem we get

$$\begin{aligned} A(x) &= \mathbf{P}(X > x) = \frac{\mathbb{E}\left[I\{X > x\} e^{\rho X}\right]}{m} = \frac{1}{m}\int_x^\infty e^{\rho y} p_X(y) dy \\ &= \frac{1}{m}\int_x^\infty \frac{l_0(y)dy}{y^{\beta+1}} \sim \frac{1}{m\beta}\frac{l_0(x)}{x^\beta} = \frac{l(x)}{x^\beta}, \end{aligned} \tag{17}$$

where $l(x)$ is a function slowly varying at infinity. Thus, the random variable $X$ under the measure $\mathbf{P}$ does not satisfy the Cramer condition and has finite variance.

The density of $X$ under $\mathbf{P}$ is

$$\mathbf{p}_X(x) = -A'(x) = \frac{1}{m}\frac{l_0(x)}{x^{\beta+1}}$$

and it satisfies (see Theorem 1.5.2 p22 in [8]) for each $M \geq 0$ and $\epsilon(x) \to 0$ as $x \to 0$,

$$\frac{\mathbf{p}_X(x + t\epsilon(x)x)}{\mathbf{p}_X(x)} \overset{x\to\infty}{\longrightarrow} 1, \tag{18}$$

uniformly with respect to $t \in [-M, M]$. In particular,

$$A(x+\Delta) - A(x) = -\frac{\Delta\beta A(x)}{x}(1 + o(1)) \tag{19}$$

as $x \to \infty$ and setting

$$b_n = \beta\frac{A(an)}{an},$$

we have

$$b_n^{-1}\mathbf{p}_X(an+t\sqrt{n}) \overset{n\to\infty}{\longrightarrow} 1, \tag{20}$$

uniformly with respect to $t \in [-M, M]$.

Let $\mathbf{\Phi} = \{\Phi\}$ be the metric space of the Laplace transforms $\Phi(\lambda) = \int_0^\infty e^{-\lambda u}\mathcal{L}(du)$, $\lambda \in [0,\infty)$, of the laws from $\mathfrak{L}$ endowed with the metric

$$d(\Phi_1, \Phi_2) = \sup_{2^{-1}\leq\lambda\leq 2} |\Phi_1(\lambda) - \Phi_2(\lambda)|.$$

Since the Laplace transform of the distribution of a nonnegative random variable is completely determined by its values on any interval of the positive half-line, convergence $\Phi_n \to \Phi$ as $n\to\infty$ in metric $d$ is equivalent to weak convergence $\mathcal{L}_n \overset{w}{\to} \mathcal{L}$ of the respective probability measures.
From now on, to avoid confusions we agree to use $P$ and $E$ for the symbols of probability and expectation in the case when the respective distributions are not associated with the measures $\mathbb{P}$ or $\mathbf{P}$.
Let $\mathfrak{F}$= $\{f(s)\}$ be the set of all probability generating functions of integer-valued random variables $\eta \geq 0$, i.e. $f(s) = E[s^\eta]$ and let $\mathbf{\Phi}^{(f)} \subset \mathbf{\Phi}$ be the closure (in metric $d$) of the set of all Laplace transforms of the form

$$\Phi(\lambda; f) = f\left(\exp\left\{-\lambda/f'(1)\right\}\right), \quad f \in \mathfrak{F}.$$

The probability measure $\mathbf{P}$ on $\mathfrak{N}$ generates a natural probability measure on the metric space $\mathbf{\Phi}^{(f)}$ which we denote by the same symbol $\mathbf{P}$.
Introduce a sequence of probability measures on $\mathbf{\Phi}^{(f)}$ by the equality

$$\mathbf{P}^{[x]}(\cdot) = \mathbf{P}\left(\cdot \mid f'(1;\mathfrak{e}) = e^x\right).$$

With this new probability measure, Hypothesis B is now equivalent to

**Hypothesis B'**. There exists a measure $\mathbf{P}^*(\cdot)$ on $\mathbf{\Phi}^{(f)}$ (with the support on $\Phi(\lambda) : \Phi(0) = 1$, $\Phi(\lambda) < 1$, $\lambda > 0$) such that, as $x \to \infty$

$$\mathbf{P}^{[x]} \Longrightarrow \mathbf{P}^*.$$

In the other words, Hypothesis B' means that there exists a (random) a.s. continuous on $[0,\infty)$ function $g(\cdot)$ with values in $\mathbf{\Phi}^{(f)}$ such that, for every continuous bounded functional $H$ on $\mathbf{\Phi}^{(f)}$

$$\lim_{x\to\infty} \mathbf{E}^{[x]}[H(\Phi)] = \mathbf{E}^*[H(g)]. \tag{21}$$

Since, for any fixed $\lambda \geq 0$ the functional $H_\lambda(\Phi) = \Phi(\lambda)$ is continuous on $\mathbf{\Phi}^{(f)}$, we have for $y = e^x$

$$\lim_{y\to\infty} \mathbf{E}\left[f(e^{-\lambda/y};\mathfrak{e}) \mid f'(1;\mathfrak{e}) = y\right] = \mathbf{E}^*[g(\lambda)], \quad \lambda \in [0,\infty) \tag{22}$$

and $\mathbf{E}^*[g(0)] = 1, \mathbf{E}^*[g(\lambda)] < 1$ if $\lambda > 0$. The prelimiting functions at the left-hand side of (22) have the form

$$\mathbf{E}\left[f(e^{-\lambda/y};\mathfrak{e}) \mid f'(1;\mathfrak{e}) = y\right] = \mathbf{E}\left[e^{-\lambda\xi(\mathfrak{e})/y} \mid f'(1;\mathfrak{e}) = y\right]$$

and, therefore, are the Laplace transforms of the distributions of some random variables. Hence, by the continuity theorem for Laplace transforms there exists a proper nonnegative random variable $\theta$ such that

$$\lim_{y\to\infty} \mathbf{E}\left[f(e^{-\lambda/y};\mathfrak{e})\mid f'(1;\mathfrak{e}) = y\right] = \mathbf{E}^*\left[e^{-\lambda\theta}\right], \quad \lambda \in [0,\infty).$$

Let now

$$h(s) = E\left[s^{\upsilon}\right] = \sum_{k=0}^{\infty} h_k s^k, \ h(1) = 1$$

be the (deterministic) probability generating function of the nonnegative integer-valued random variable $\upsilon$. Since, for any fixed $\lambda \geq 0$ the functional $H_{\lambda,h}(\Phi) = h(\Phi(\lambda))$ is continuous on $\mathbf{\Phi}^{(f)}$, we have

$$\lim_{y\to\infty} \mathbf{E}\left[h\left(f(e^{-\lambda/y};\mathfrak{e})\right) \mid f'(1;\mathfrak{e}) = y\right] = \mathbf{E}^*\left[h\left(g(\lambda)\right)\right], \quad \lambda \in [0,\infty). \tag{23}$$

The prelimiting and limiting functions are monotone and continuous on $[0,\infty)$. Therefore, convergence in (23) is uniform in $\lambda \in [0,\infty)$

Further, denoting by $\xi_i(\mathfrak{e}), i = 1,2,...$ independent copies of $\xi(\mathfrak{e})$ we get

$$\begin{aligned}
\mathbf{E}\left[h\left(f(e^{-\lambda/y};\mathfrak{e})\right) \mid f'(1;\mathfrak{e}) = y\right] &= \sum_{k=0}^{\infty} h_k \mathbf{E}\left[f^k(e^{-\lambda/y};\mathfrak{e}) \mid f'(1;\mathfrak{e}) = y\right] \\
&= \sum_{k=0}^{\infty} h_k \mathbf{E}\left[\exp\left\{-\frac{\lambda}{y}\sum_{i=1}^{k}\xi_i(\mathfrak{e})\right\} \;\Big|\; f'(1;\mathfrak{e}) = y\right] \\
&= \mathbf{E}\left[\exp\left\{-\frac{\lambda}{y}\Xi\right\} \;\Big|\; f'(1;\mathfrak{e}) = y\right],
\end{aligned}$$

where

$$\Xi(\mathfrak{e}) = \sum_{i=1}^{\upsilon} \xi_i(\mathfrak{e}).$$

Thus, similarly to the previous arguments there exists a proper random variable $\Theta$ such that

$$\lim_{y\to\infty} \mathbf{E}\left[\exp\left\{-\frac{\lambda}{y}\Xi(\mathfrak{e})\right\} \Big| f'(1;\mathfrak{e}) = y\right] = \mathbf{E}^*\left[e^{-\lambda\Theta}\right], \quad \lambda \in [0,\infty). \tag{24}$$

As above, this convergence is uniform with respect to $\lambda \in [0,\infty)$.

## 4.2 Some useful results on random walks

We pick here from [7] several results on random walks with negative drift and heavy tails useful for the forthcoming proofs. Recall that $b_n = \beta A(an)/(an)$, and introduce three important random variables

$$M_n \;=\; \max(S_1, \dots, S_n)\ , \quad L_n \;=\; \min(S_1, \dots, S_n),$$

and

$$\tau_n = \min\{0 \le k \le n : S_k = L_n\}$$

and two right-continuous functions $U : \mathbb{R} \to \mathbb{R}_0 = \{x \ge 0\}$ and $V : \mathbb{R} \to \mathbb{R}_0$ given by

$$U(x) \;=\; 1 + \sum_{k=1}^{\infty} \mathbf{P}\left(-S_k \le x, M_k < 0\right)\ , \quad x \ge 0,$$

$$V(x) \;=\; 1 + \sum_{k=1}^{\infty} \mathbf{P}\left(-S_k > x, L_k \ge 0\right)\ , \quad x \le 0,$$

and 0 elsewhere. In particular $U(0) = V(0) = 1$. It is well-known that $U(x) = O(x)$ for $x \to \infty$. Moreover, $V(-x)$ is uniformly bounded in $x$ in view of $\mathbf{E}X < 0$.

With this notation in hands we recall the following result established in Lemma 7 of [7].

**Lemma 5** *Assume that* $\mathbf{E}[X] < 0$ *and that* $A(x)$ *meets condition* (19)*. Then, for any* $\lambda > 0$ *as* $n \to \infty$

$$\mathbf{E}\left[e^{\lambda S_n}; \tau_n = n\right] = \mathbf{E}\left[e^{\lambda S_n}; M_n < 0\right] \sim b_n \int_0^{\infty} e^{-\lambda z} U(z)\, dz \tag{25}$$

*and*

$$\mathbf{E}\left[e^{-\lambda S_n}; \tau > n\right] = \mathbf{E}\left[e^{-\lambda S_n}; L_n \ge 0\right] \sim b_n \int_0^{\infty} e^{-\lambda z} V(-z)\, dz. \tag{26}$$

Moreover from (19) and (20) in [7], we know that for $\lambda > 0$

$$b_n^{-1}\mathbf{E}[e^{\lambda S_n}; M_n < 0\,,\, S_n < -x] \;\to\; \int_x^{\infty} e^{-\lambda z} U(z)\, dz, \tag{27}$$

$$b_n^{-1}\mathbf{E}[e^{-\lambda S_n}; L_n \ge 0\,,\, S_n > x] \;\to\; \int_x^{\infty} e^{-\lambda z} V(-z)\, dz. \tag{28}$$

and gathering Lemmas 9,10,11 in [7] yields

**Lemma 6** *If* $\mathbf{E}[X] = -a < 0$ *and condition (19) is valid then*
*(i) there exists* $\delta_0 \in (0, 1/4)$ *such that for* $an/2 - u \ge\ M$ *and all* $\delta \in (0, \delta_0)$ *and* $k \in \mathbb{Z}$

$$\mathbf{P}_u(\max_{1 \le j \le n} X_j \le \delta an,\ S_n \ge k) \le \varepsilon_M(k) n^{-\beta-1},$$

*where* $\varepsilon_M(k) \downarrow_{M\to\infty} 0$. *Moreover, for any fixed* $l$ *and* $\delta \in (0,1)$

$$\lim_{J\to\infty} \limsup_{n\to\infty} b_n^{-1}\mathbf{P}\left(L_n \geq -N,\ \max_{J\leq j\leq n} X_j \geq \delta an,\ S_n \in [l, l+1)\right) = 0.$$

*(ii) for any fixed* $\delta \in (0,1)$ *and* $K \geq 0$,

$$\lim_{M\to\infty} \limsup_{n\to\infty} b_n^{-1}\mathbf{P}\left(\delta an \leq X_1 \leq an - M\sqrt{n} \text{ or } X_1 \geq an + M\sqrt{n}; |S_n| \leq K\right) = 0.$$

*(iii) for each fixed* $\delta > 0$ *and* $J \geq 2$

$$\lim_{n\to\infty} b_n^{-1}\mathbf{P}\left(\cup_{i\neq j}^J \{X_i \geq \delta an, X_j \geq \delta an\}\right) = 0.$$

Combining the limit for $J \to \infty$ in (i) with (iii), we get that for any fixed $N, K \geq 0$

$$\lim_{n\to\infty} b_n^{-1}\mathbf{P}\left(\cup_{i\neq j}^n \{X_i \geq \delta an, X_j \geq \delta an\}; L_n \geq -N, |S_n| \leq K\right) = 0. \tag{29}$$

# 5 Proofs

In this section we use the notation

$$\mathbf{E}_{\mathfrak{e}}[\cdot] = \mathbf{E}[\cdot \mid \mathcal{E}], \quad \mathbf{P}_{\mathfrak{e}}(\cdot) = \mathbf{P}(\cdot \mid \mathcal{E})$$

i.e., consider the expectation and probability given the environment $\mathcal{E}$. Our aim is to prove the following statement.

**Lemma 7** *If Hypotheses A and B are valid then there exists a constant* $C_0 > 0$ *such that, as* $n \to \infty$

$$\mathbb{P}(Z_n > 0) \sim C_0 m^n \beta \frac{\mathbf{P}(X > an)}{an} = C_0 m^n b_n. \tag{30}$$

We recall from the discussion in Preliminaries that Hypotheses A and B (or B') ensure that there exists $g(\lambda)$ a.s. continuous on $[0,\infty)$ with $g(0) = 1$ and with $\mathbf{E}[g(\lambda)] < 1, \lambda > 0$, such that for every continuous bounded function $H$ on $[0,1]$

$$\lim_{y\to\infty} \sup_{\lambda\geq 0} \left|\mathbf{E}\left[H(f(e^{-\lambda/y})) \mid f'(1) = y\right] - \mathbf{E}^*[H(g(\lambda))]\right| = 0. \tag{31}$$

Making the change of measure in accordance with (14) and (15) we see that it is necessary to show that, as $n \to \infty$

$$\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}(Z_n > 0) e^{-\rho S_n}\right] \sim C_0 b_n. \tag{32}$$

The proof of this fact is conducted into several steps which we split into subsections.

## 5.1 Time of the minimum of $S$

First, we prove that the contribution to $\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n}\right]$ may be of order $b_n$ only if the minimal value of $S$ within the interval $[0,n]$ is attained at the beginning or at the end of this interval. To this aim we use, as earlier, the notation $\tau_n=\min\{0\le k\le n: S_k=L_n\}$ and show that the following statement is valid.

**Lemma 8** *Given Hypotheses A and B we have*

$$\lim_{M\to\infty}\lim_{n\to\infty}b_n^{-1}\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n\in[M,n-M]\right]=0.$$

**Proof.** In view of the estimate

$$\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)\le\min_{0\le k\le n}\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)\le\exp\left\{\min_{0\le k\le n}S_k\right\}=e^{S_{\tau_n}}$$

we have

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n\in[M,n-M]\right]\\
&\quad\le\mathbf{E}\left[e^{S_{\tau_n}-S_n};\tau_n\in[M,n-M]\right]\\
&\quad=\sum_{k=M}^{n-M}\mathbf{E}\left[e^{(1-\rho)S_k+\rho(S_k-S_n)};\tau_n=k\right]\\
&\quad=\sum_{k=M}^{n-M}\mathbf{E}\left[e^{(1-\rho)S_k};\tau_k=k\right]\mathbf{E}\left[e^{-\rho S_{n-k}};L_{n-k}\ge 0\right].
\end{aligned}\tag{33}$$

Hence, using Lemma 5 we get

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n\in[M,n-M]\right]\\
&\le\left(\sum_{k=M}^{[n/2]}+\sum_{k=[n/2]+1}^{n-M}\right)\mathbf{E}\left[e^{(1-\rho)S_k};\tau_k=k\right]\mathbf{E}\left[e^{-\rho S_{n-k}};L_{n-k}\ge 0\right]\\
&\le\frac{C_1}{n}\mathbf{P}\left(X>\frac{an}{2}\right)\sum_{k=M}^{[n/2]}\mathbf{E}\left[e^{(1-\rho)S_k};\tau_k=k\right]\\
&+\frac{C_2}{n}\mathbf{P}\left(X>\frac{an}{2}\right)\sum_{k=M}^{[n/2]}\mathbf{E}\left[e^{-\rho S_k};L_k\ge 0\right]\le\varepsilon_M b_n
\end{aligned}\tag{34}$$

where $\varepsilon_M\to 0$ as $M\to\infty$. ■

The following statement easily follows from (34) by taking $M=0$.

**Corollary 9** *Given Hypotheses A and B there exists $C\in(0,\infty)$ such that, for all $n=1,2,...$*

$$\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n}\right]\le\mathbf{E}\left[e^{S_{\tau_n}-\rho S_n}\right]\le Cb_n.$$

## 5.2 Fluctuations of the random walk $S$

Introduce the event

$$\mathcal{C}_N = \left\{ -N < S_{\tau_n} \le S_n \le N + S_{\tau_n} < N \right\}.$$

In particular, given $\mathcal{C}_N$

$$-N < S_n < N.$$

In what follows we agree to denote by $\varepsilon_N, \varepsilon_{N,n}$ or $\varepsilon_{N,K,n}$ functions of the low indices such that

$$\lim_{N\to\infty} \varepsilon_N = \lim_{N\to\infty} \limsup_{n\to\infty} |\varepsilon_{N,n}| = \lim_{N\to\infty} \limsup_{K\to\infty} \limsup_{n\to\infty} |\varepsilon_{N,K,n}| = 0,$$

i.e., the $\limsup$ (or $\lim$) are sequentially taken with respect to the indices of $\varepsilon_{\dots}$ in the reverse order. Note that the functions are not necessarily the same in different formulas or even within one and the same complicated expression.

**Lemma 10** *Given Hypotheses A and B for any fixed $k$*

$$\lim_{N\to\infty} \limsup_{n\to\infty} b_n^{-1} \mathbf{E}\left[ \mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right) e^{-\rho S_n}; \tau_n = k, \bar{\mathcal{C}}_N \right] = 0$$

*and*

$$\lim_{N\to\infty} \limsup_{n\to\infty} b_n^{-1} \mathbf{E}\left[ \mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right) e^{-\rho S_n}; \tau_n = n-k, \bar{\mathcal{C}}_N \right] = 0.$$

**Proof.** In view of (28)

$$\begin{aligned}
&\mathbf{E}\left[ \mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right) e^{-\rho S_n}; \tau_n = k, S_n - S_{\tau_n} \ge N \right] \\
&\quad \le \mathbf{E}\left[ e^{(1-\rho)S_{\tau_n}} e^{-\rho(S_n - S_{\tau_n})}; \tau_n = k, S_n - S_{\tau_n} \ge N \right] \\
&\quad\quad \le \mathbf{E}\left[ e^{-\rho S_{n-k}}; L_{n-k} \ge 0, S_{n-k} \ge N \right] \le \varepsilon_N b_n
\end{aligned}$$

where $\varepsilon_N \to 0$ as $N \to \infty$ since $\int_0^\infty \exp(-\rho z) V(-z) dz < \infty$. Further,

$$\begin{aligned}
&\mathbf{E}\left[ \mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right) e^{-\rho S_n}; \tau_n = k, S_{\tau_n} \le -N \right] \\
&\quad \le \mathbf{E}\left[ e^{(1-\rho)S_{\tau_n}} e^{-\rho(S_n - S_{\tau_n})}; \tau_n = k, S_{\tau_n} \le -N \right] \\
&\quad\quad \le e^{-(1-\rho)N} \mathbf{E}\left[ e^{-\rho S_{n-k}}; L_{n-k} \ge 0 \right] \le \varepsilon_N b_n.
\end{aligned} \tag{35}$$

This, in particular, means that

$$\mathbf{E}\left[ e^{(1-\rho)S_{\tau_n}} e^{-\rho(S_n - S_{\tau_n})}; \tau_n = k, S_n \notin (-N, N) \right] = \varepsilon_{N,n} b_n \tag{36}$$

and

$$\begin{aligned}
&\mathbf{E}\left[ \mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right) e^{-\rho S_n}; \tau_n = k \right] \\
&\quad = \mathbf{E}\left[ \mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right) e^{-\rho S_n}; \tau_n = k, S_{\tau_n} \ge -N, S_n - S_{\tau_n} \le N \right] + \varepsilon_{N,n} b_n.
\end{aligned}$$

Similarly, by (27)

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n=n-k,S_{\tau_n}\leq -N\right]\\
&\quad\leq \mathbf{E}\left[e^{(1-\rho)S_{\tau_n}}e^{-\rho(S_n-S_{\tau_n})};\tau_n=n-k,S_{\tau_n}\leq -N\right]\\
&\quad\quad\leq \mathbf{E}\left[e^{(1-\rho)S_{n-k}};\tau_{n-k}=n-k,S_{n-k}\leq -N\right]\\
&\quad\quad= \mathbf{E}\left[e^{(1-\rho)S_{n-k}};M_{n-k}<0,S_{n-k}\leq -N\right]=\varepsilon_{N,n}b_n
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n=n-k,S_n-S_{\tau_n}\geq N\right]\\
&\quad\leq \mathbf{E}\left[e^{(1-\rho)S_{\tau_n}}e^{-\rho(S_n-S_{\tau_n})};\tau_n=n-k,S_n-S_{\tau_n}\geq N\right]\\
&\quad\quad\leq e^{-\rho N}\mathbf{E}\left[e^{(1-\rho)S_{n-k}};\tau_{n-k}=n-k\right]\\
&\quad\quad= e^{-\rho N}\mathbf{E}\left[e^{(1-\rho)S_{n-k}};M_{n-k}<0\right]=\varepsilon_{N,n}b_n.
\end{aligned}$$

As a result we get

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n=n-k\right]\\
&\quad=\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n=n-k,S_{\tau_n}\geq -N,S_n-S_{\tau_n}\leq N\right]+\varepsilon_{N,n}b_n.
\end{aligned}$$

This completes the proof of the lemma. ■

Lemmas 8 and 10 easily imply the following statement:

**Corollary 11** *Under Hypotheses A and B*

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n}\right]\\
&\quad=\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};|S_n|<N;\tau_n\in[0,M]\cup[n-M,n]\right]+\varepsilon_{N,M,n}b_n\\
&\quad\quad=\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};|S_n|<N\right]+\varepsilon_{N,n}b_n\\
&\quad\quad\quad=\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};S_{\tau_n}\geq -N,S_n<N\right]+\tilde{\varepsilon}_{N,n}b_n
\end{aligned}\tag{37}$$

*where*

$$\lim_{N\to\infty}\limsup_{M\to\infty}\limsup_{n\to\infty}|\varepsilon_{N,M,n}|=\lim_{N\to\infty}\limsup_{n\to\infty}\left(|\varepsilon_{N,n}|+|\tilde{\varepsilon}_{N,n}|\right)=0.$$

## 5.3 Asymptotic of the survival probability

In this section we investigate in detail the properties of the survival probability for the processes meeting Hypotheses A and B. As we know (see (15)) this probability is expressed as

$$\mathbb{P}\left(Z_n>0\right)=m^n\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n}\right].$$

We wish to show that $\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n}\right]$ is of order $b_n$ as $n\to\infty$.

First we get rid of the trajectories giving the contribution of the order $o(b_n)$ to the quantity in question. Let

$$\mathcal{D}_N(j) = \{-N < S_{\tau_n} \le S_n < N,\ X_j \ge \delta a n\}.$$

**Lemma 12** *If Hypotheses A and B are valid then there exists $\delta \in (0, 1/4)$ such that*

$$\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right)\exp(-\rho S_n)\right] = \sum_{j=1}^{J} \mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right)\exp(-\rho S_n); \mathcal{D}_N(j)\right] + \varepsilon_{N,J,n} b_n. \tag{38}$$

**Proof.** In view of Corollary 11, we just need to prove that

$$\begin{aligned}\mathbf{E}&\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right)e^{-\rho S_n}; S_{\tau_n} \ge -N, S_n < N\right] \\ &= \sum_{j=1}^{J} \mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right)\exp(-\rho S_n); \mathcal{D}_N(j)\right] + \varepsilon_{N,J,n} b_n.\end{aligned} \tag{39}$$

From the estimate

$$\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right)\exp(-\rho S_n) \le \exp(S_{\tau_n} - \rho S_n) = \exp((1-\rho)S_{\tau_n} - \rho(S_n - S_{\tau_n})) \le 1 \tag{40}$$

we deduce by Lemma 6 (i) that

$$\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right)e^{-\rho S_n}; S_{\tau_n} \ge -N, S_n < N, \max_{0\le j\le n} X_j < \delta a n\right] = \varepsilon_{N,n} b_n$$

for $\delta \in (0, \delta_0)$ and

$$\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right)e^{-\rho S_n}; S_{\tau_n} \ge -N, S_n < N, \max_{J\le j\le n} X_j \ge \delta a n\right] = \varepsilon_{N,J,n} b_n.$$

Thus,

$$\begin{aligned}\mathbf{E}&\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right)e^{-\rho S_n}; S_{\tau_n} \ge -N, S_n < N\right] \\ &= \mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right)e^{-\rho S_n}; S_{\tau_n} \ge -N, S_n < N, \max_{0\le j\le J} X_j \ge \delta a n\right] + \varepsilon_{N,J,n} b_n.\end{aligned}$$

Finally thanks to Lemma 6(iii), there is only one big jump (before $J$),i.e.

$$\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right)e^{-\rho S_n}; S_{\tau_n} \ge -N, S_n < N, \cup_{i\ne j}^{J}\left\{X_i \ge \delta a n, X_j \ge \delta a n\right\}\right] = \varepsilon_{N,J,n} b_n.$$

It yields (39) and ends up the proof. ∎

Now we fix $j \in [1, J]$ and investigate the quantity

$$\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right)\exp(-\rho S_n); \mathcal{D}_N(j)\right].$$

First, we check that $S_{j-1}$ is bounded on the event we focus on.

**Lemma 13** *If Hypotheses A and B are valid then, for every fixed $j$*

$$\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)\exp(-\rho S_n);|S_{j-1}|\geq N,X_j\geq\delta an\right]=\varepsilon_{N,n}b_n.$$

**Proof.** First observe that

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)\exp(-\rho S_n);S_{j-1}\leq -N,X_j\geq\delta an\right]\\
&\quad\leq\mathbf{E}\left[\exp((1-\rho)S_{\tau_n}-\rho(S_n-S_{\tau_n}));S_{j-1}\leq -N,X_j\geq\delta an\right]\\
&\quad\leq\mathbf{E}\left[\exp((1-\rho)S_{\tau_n});S_{j-1}\leq -N,X_j\geq\delta an\right]\\
&\quad\leq\mathbf{E}\left[\exp(-(1-\rho)N);X_j\geq\delta an\right]\\
&\quad=\exp(-(1-\rho)N)\mathbf{P}(X\geq\delta an)=\varepsilon_{N,n}b_n.
\end{aligned}$$

Further, taking $\gamma\in(0,1)$ such that $\gamma\beta>1$, we get

$$\begin{aligned}
&\mathbf{E}\left[\exp(S_{\tau_n}-\rho S_n);S_{j-1}\geq n^{\gamma},X_j\geq\delta an\right]\leq\mathbf{P}(S_{j-1}\geq n^{\gamma})\mathbf{P}(X\geq\delta an)\\
&\quad\leq j\mathbf{P}(X\geq n^{\gamma}/j)\mathbf{P}(X\geq\delta an)\sim\frac{j^{\beta+1}}{n^{\gamma\beta}}l(n^{\gamma})\mathbf{P}(X\geq\delta an)=\varepsilon_n b_n. \qquad (41)
\end{aligned}$$

Consider now the situation $S_{j-1}\in[N,n^{\gamma}]$, $j\geq 2$ and write

$$\begin{aligned}
&\mathbf{E}\left[\exp(S_{\tau_n}-\rho S_n);S_{j-1}\in[N,n^{\gamma}],X_j\geq\delta an\right]\\
&\quad=\int_N^{n^{\gamma}}\int_{-\infty}^0\mathbf{P}(S_{j-1}\in dy,L_{j-1}\in dz)H_{n,\delta}(y,z),
\end{aligned}$$

where

$$\begin{aligned}
H_{n,\delta}(y,z)&=\int_{\delta an}^{\infty}\mathbf{P}(X\in dt)\int_{-\infty}^0\int_v^{\infty}\mathbf{P}_{y+t}(L_{n-j}\in dv,S_{n-j}\in dw)e^{z\wedge v}e^{-\rho w}\\
&=\int_{\delta an+y}^{\infty}\mathbf{P}(X\in dt-y)\int_{-\infty}^0\int_v^{\infty}\mathbf{P}_{t}(L_{n-j}\in dv,S_{n-j}\in dw)e^{z\wedge v}e^{-\rho w}.
\end{aligned}$$

By our conditions $\mathbf{P}(X\in dt-y)=\mathbf{P}(X\in dt)\left(1+o\left(1\right)\right)$ uniformly in $y\in[0,n^{\gamma}]$ and $t\geq\delta an$. Thus, for all sufficiently large $n$

$$\begin{aligned}
H_{n,\delta}(y,z)&\leq 2\int_{\delta an}^{\infty}\mathbf{P}(X\in dt)\int_{-\infty}^0\int_v^{\infty}\mathbf{P}_{t}(L_{n-j}\in dv,S_{n-j}\in dw)e^{z\wedge v}e^{-\rho w}\\
&\leq 2\int_{\delta an}^{\infty}\mathbf{P}(X\in dt)\int_{-\infty}^0\int_v^{\infty}\mathbf{P}_{t}(L_{n-j}\in dv,S_{n-j}\in dw)e^{v}e^{-\rho w}\\
&=2\int_{\delta an}^{\infty}\mathbf{P}(X\in dt)\mathbf{E}_t\left[e^{S_{\tau_{n-j}}-\rho S_{n-j}}\right]\\
&\leq 2\mathbf{E}_0\left[e^{S_{\tau_{n-j+1}}-\rho S_{n-j+1}};X_1\geq\delta an\right]=2H_{n,\delta}(0,0).
\end{aligned}$$

By integrating this inequality we get for sufficiently large $n$

$$
\begin{aligned}
&\int_N^{n^\gamma}\int_{-\infty}^0 \mathbf{P}(S_{j-1}\in dy, L_{j-1}\in dz)H_{n,\delta}(y,z)\\
&\qquad\le 2\int_N^{n^\gamma}\int_{-\infty}^0 \mathbf{P}(S_{j-1}\in dy, L_{j-1}\in dz)H_{n,\delta}(0,0)\\
&\qquad\le 2\mathbf{P}(S_{j-1}\ge N)\mathbf{E}_0\left[e^{S_{\tau_{n-j+1}}-\rho S_{n-j+1}}; X_1\ge \delta an\right].
\end{aligned}
$$

Since

$$
\begin{aligned}
b_n^{-1}H_{n,\delta}(0,0) &= b_n\mathbf{E}\left[e^{S_{\tau_{n-j+1}}-\rho S_{n-j+1}}; X_1\ge \delta an\right]\\
&\le b_n^{-1}\mathbf{E}\left[e^{S_{\tau_{n-j+1}}-\rho S_{n-j+1}}\right] = O(1)
\end{aligned}
$$

as $n\to\infty$ (see Corollary 9) and $\mathbf{P}(S_{j-1}\ge N)\to 0$ as $N\to\infty$, we obtain

$$
\mathbf{E}\left[\exp(S_{\tau_n}-\rho S_n);\, S_{j-1}\in[N,n^\gamma], X_j\ge\delta n\right] = \varepsilon_{N,n}b_n. \tag{42}
$$

Combining (41) and (42) proves the lemma. ■

**Lemma 14** *Given Hypotheses A and B we have for each fixed $j$*

$$
\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)\exp(-\rho S_n); |S_n-S_{j-1}|>K, X_j\ge\delta an\right] = \varepsilon_{K,n}(j)b_n.
$$

**Proof.** We know from Lemma 13 that only the values $S_{j-1}\le N$ for sufficiently large but fixed $N$ are of importance. Thus, we just need to prove that for fixed $N$

$$
\mathbf{E}\left[e^{S_{\tau_n}-\rho S_n}; S_{j-1}\le N, |S_n-S_{j-1}|>K, X_j\ge\delta an\right] = \varepsilon_{N,K,n}(j)b_n
$$

where $\lim_{K\to\infty}\limsup_{n\to\infty}|\varepsilon_{N,K,n}(j)|=0$. To this aim we set $L_{j,n}=\min\{S_k-S_{j-1}: j-1\le k\le n\}$ and, using the inequality $S_{\tau_n}\le S_{j-1}+L_{j,n}$, deduce the estimate

$$
\begin{aligned}
&\mathbf{E}\left[e^{S_{\tau_n}-\rho S_n}; S_{j-1}\le N, |S_n-S_{j-1}|>K, X_j\ge\delta an\right]\\
&\quad\le \mathbf{E}\left[e^{S_{j-1}+L_{j,n}-\rho(S_n-S_{j-1})-\rho S_{j-1}}; S_{j-1}\le N, |S_n-S_{j-1}|>K\right]\\
&\quad= \mathbf{E}\left[e^{(1-\rho)S_{j-1}}; S_{j-1}\le N\right]\mathbf{E}\left[e^{L_{j,n}-\rho(S_n-S_{j-1})}; |S_n-S_{j-1}|>K\right].
\end{aligned}
$$

We conclude with $\mathbf{E}\left[e^{(1-\rho)S_{j-1}}; S_{j-1}\le N\right]<\infty$ and we can now control the term

$$
\mathbf{E}\left[e^{L_{j,n}-\rho(S_n-S_{j-1})}; |S_n-S_{j-1}|>K\right] = \mathbf{E}\left[e^{S_{\tau_{n-j+1}}-\rho S_{n-j+1}}; |S_{n-j+1}|>K\right]
$$

by $\varepsilon_{K,n}b_n$. Indeed it is now exactly the term controlled in a similar situation in (36). ■

We give the last technical lemma.

**Lemma 15** *Assume that $g$ is a random function which satisfies (31). Then for every (deterministic) probability generating function $h(s)$ and every $\varepsilon > 0$ there exists $\kappa > 0$ such that*

$$\left| \mathbf{E}\left[1 - h(g(e^{v}w))\right] - \mathbf{E}\left[1 - h(g(e^{v'}w))\right] \right| \leq h'(1)\varepsilon$$

*for $|v - v'| \leq \kappa, w \in [0, 2]$.*

**Proof.** Clearly,

$$\left| \mathbf{E}\left[1 - h(g(e^{v}w))\right] - \mathbf{E}\left[1 - h(g(e^{v'}w))\right] \right| \leq h'(1)\mathbf{E}\left[|g(e^{v'}w) - g(e^{v}w)|\right].$$

We know that $0 \leq g(\lambda) \leq 1$ for all $\lambda \in [0, \infty)$, $g(\lambda)$ is nonincreasing with respect to $\lambda$ a.s. and has a finite limit as $\lambda \to \infty$. Therefore, $g(\lambda)$ is a.s. uniformly continuous on $[0, \infty)$ implying that a.s.

$$\lim_{\kappa \to 0} \sup_{|v-v'| \leq \kappa, w \in [0,2]} |g(e^{v'}w) - g(e^{v}w)| = 0.$$

Hence, by the bounded convergence theorem

$$\sup_{|v-v'| \leq \kappa, w \in [0,2]} \mathbf{E}\left[|g(e^{v'}w) - g(e^{v}w)|\right] \leq \mathbf{E}\left[\sup_{|v-v'| \leq \kappa, w \in [0,2]} |g(e^{v'}w) - g(e^{v}w)|\right]$$

goes to zero as $\kappa \to 0$, which ends up the proof. ∎

Let $\sigma^2 = VarX$ , $S_{n,j} = S_n - S_j$, $0 \leq j \leq n$, and

$$G_{n,j} = -\frac{S_{n,j} + an}{\sigma\sqrt{n}}.$$

Using the notation (11), we write

$$\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right) = 1 - f_{0,n}(0)$$

put $\mathbf{X}_{j,n} = (X_{j+1}, \cdots, X_n)$, $\mathbf{X}_{n,j} = (X_n, \cdots, X_{j+1})$, and set

$$Y_j = F(S_0, \mathbf{S}_{0,j-1}), \qquad Y_{j,n} = F_n(S_n - S_{j-1}, \mathbf{X}_{j,n}), \qquad Y_{n,j} = F_n(S_n - S_{j-1}, \mathbf{X}_{n,j}),$$

where $F, F_n$ are positive equibounded measurable functions.

Since $f_{j,n}$ is distributed as $f_{n,j}$, we write

$$\begin{aligned}
&\mathbf{E}[Y_j Y_{j,n} \mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right) e^{-\rho S_n}; X_j \geq \delta an] \\
&\quad = \mathbf{E}[Y_j Y_{j,n}\left(1 - f_{0,n}(0)\right) e^{-\rho S_n}; X_j \geq \delta an] \\
&\quad = \mathbf{E}[Y_j Y_{n,j}(1 - f_{0,j-1}(f_j(f_{n,j}(0))))e^{-\rho S_n}; X_j \geq \delta an] \\
&\quad = \mathbf{E}[Y_j e^{-\rho S_{j-1}} Y_{n,j}\left(1 - f_{0,j-1}(f_j(f_{n,j}(0)))\right) e^{-\rho(S_n - S_{j-1})}; X_j \geq \delta an] \\
&\quad = \mathbf{E}[Y_j e^{-\rho S_{j-1}} Y_{n,j}\left(1 - f_{0,j-1}(f_j(1 - e^{S_{n,j}} W_{n,j}))\right) e^{-\rho S_{n,j-1}}; X_j \geq \delta an]
\end{aligned}$$

where $W_{n,j}$ were defined in (12). Our aim is to obtain an approximation to this expression.
To simplify notation we let

$$\bar{h}\,(s) = 1 - h(s)$$

for a probability generating function $h\,(s)$. For fixed positive $M$ and $K$, we set

$$B_{j,n} = \{S_{n,j} \in [-K,K],\, X_j - na \in [-M\sqrt{n}, M\sqrt{n}]\},$$

and define

$$F_{n,j}(h,K,M) = \mathbf{E}\left[e^{-\rho(S_{n,j-1})} Y_{n,j} \bar{h}\left(f_j\left(\exp\left\{-e^{S_{n,j}} W_{n,j}\right\}\right)\right) ; B_{j,n}\right].$$

We now introduce a random function $g_j$ on the probability space $(\Omega, \mathbf{P})$, whose distribution is specified by $\mathbf{P}^*$ (i.e. $\mathbf{E}(H(g_j)) = \mathbf{E}^*(H(g))$ for any bounded $H$). Moreover we choose $g_j$ such that $g_j$ is independent of $(f_k : k \neq j)$. As we have mentioned, it is always possible by extending the initial probability space if required. We denote $Y_{n,j}(v) = F_n(v, \mathbf{X}_{n,j})$ and consider

$$O_{n,j}(h,K,M) = \int_{-K}^{K} e^{-\rho v} dv \mathbf{E}\left[Y_{n,j}(v)\bar{h}\left(g_j\left(e^v W_{n,j}\right)\right) ; \sigma G_{n,j} \in [-M,M]\right]$$

where $g_j$ is independent of $(S_k : k \geq 0)$ and $(f_k : k \neq j)$.

**Lemma 16** *For all $K, M \geq 0$ and and any probability generating function $h$ we have*

$$\lim_{n\to\infty} |b_n^{-1} F_{n,j}(h,K,M) - O_{n,j}(h,K,M)| = 0$$

**Proof.** Let $\mathcal{F}_{j,n}$ be the $\sigma$-algebra generated by the random variables

$$(f_k, X_k), \quad k = 1, 2, ..., j-1, j+1, ..., n$$

and

$$V(y, \mathbf{X}_{j,n}) = e^{-\rho y} F_n\left(y; \mathbf{X}_{n,j}\right) 1_{\{|y| \in [-K,K]\}}.$$

Using the uniform convergence (20), the change of variables $t = (x_j - an - M\sqrt{n})/\sqrt{n}$ ensures that,

$$\begin{aligned}
&b_n^{-1} F_{n,j}(h,K,M) \\
&\quad = b_n^{-1}\mathbf{E}\bigg[\int_{an-M\sqrt{n}}^{an+M\sqrt{n}} V(S_{n,j}+x_j, \mathbf{X}_{n,j}) \\
&\qquad\qquad \times \mathbf{E}\left[\bar{h}\left(f_j\left(\exp\left\{-e^{S_{n,j}} W_{n,j}\right\}\right)\right) | \mathcal{F}_{j,n}; X_j = x_j\right] \mathbf{p}_{X_j}(x_j) dx_j\bigg] \\
&\quad \sim \mathbf{E}\bigg[\int_{an-M\sqrt{n}}^{an+M\sqrt{n}} V(S_{n,j}+x_j, \mathbf{X}_{n,j}) \\
&\qquad\qquad \times \mathbf{E}\left[\bar{h}\left(f_j\left(\exp\left\{-e^{S_{n,j}} W_{n,j}\right\}\right)\right) | \mathcal{F}_{j,n}; X_j = x_j\right] dx_j\bigg],
\end{aligned}$$

when $n \to \infty$. Moreover, the uniform convergence in (23) with respect to any compact set of $\lambda$ from $[0, \infty)$ ensures that, uniformly for $|x - an| \leq Mn^{1/2}$, $w \in [0, 2]$ and $|v| \leq K$ we have

$$|\mathbf{E}\left[\bar{h}\left(f_j\left(\exp\left(-e^v w\right)\right)\right) \mid X_j = x\right] - \mathbf{E}\left[\bar{h}(g_j(e^v w))\right]| \leq \varepsilon_n.$$

Denoting $\mathcal{F}^*_{j,n}$ the $\sigma$-algebra generated by the random variables

$$X_k, \qquad k = 1, 2, ..., j-1, j+1, ..., n$$

we get, as $n \to \infty$, with $\mathbf{x}_{n,j} = (x_n, \ldots, x_{j+1})$,

$$\begin{aligned}
&b_n^{-1} F_{n,j}(h, K, M) \\
&\quad \sim \mathbf{E}\left[\int_{an - M\sqrt{n}}^{an + M\sqrt{n}} V(S_{n,j} + x_j, \mathbf{X}_{n,j}) \mathbf{E}\left[\bar{h}(g_j(e^{S_{n,j} + x_j} W_{n,j})) | \mathcal{F}^*_{j,n}\right] dx_j\right] \\
&\quad = \mathbf{E}\left[\int_{an - M\sqrt{n}}^{an + M\sqrt{n}} V(S_{n,j} + x_j, \mathbf{X}_{n,j}) \bar{h}(g_j(e^{S_{n,j} + x_j} W_{n,j})) dx_j\right] \\
&\quad \sim \int_{an - M\sqrt{n}}^{an + M\sqrt{n}} dx_j \int_{|\mathbf{x}_{n,j-1}| \leq K} V\left(\left|\mathbf{x}_{n,j-1}\right|, \mathbf{x}_{n,j}\right) \\
&\qquad\qquad \times \mathbf{E}\left[\bar{h}(g_j(e^{|\mathbf{x}_{n,j-1}|} W_{n,j}) | \mathbf{X}_{n,j} = \mathbf{x}_{n,j}\right] \prod_{i=j+1}^{n} \mathbf{p}_{X_i}\left(x_i\right) dx_i.
\end{aligned}$$

Making the change of variables

$$v = |\mathbf{x}_{n,j-1}| = x_n + x_{n-1} + ... + x_j; \qquad z_i = x_i, \quad i = j+1, ..., n$$

and setting

$$D_{n,j}\left(K, M\right) = \left\{|v| \leq K, |v - x_{j+1} - x_{j+2} - ... - x_n + an| \leq M\sqrt{n}\right\},$$

we arrive at

$$\begin{aligned}
&b_n^{-1} F_{n,j}(h, K, M) \\
&\quad \sim \int_{D_{n,j}(K,M)} e^{-\rho v} F_n(v, \mathbf{x}_{n,j}) \mathbf{E}\left[\bar{h}(g_j(e^v W_{n,j}) | \mathbf{X}_{n,j} = \mathbf{x}_{n,j}\right] \prod_{i=j+1}^{n} \mathbf{p}_{X_i}\left(x_i\right) dx_i dv \\
&\quad \sim \int_{|v| \leq K} e^{-\rho v} \mathbf{E}\left[Y_{n,j}\left(v\right) \bar{h}(g_j(e^v W_{n,j}); \sigma G_{n,j} \in [-M, M]\right] dv.
\end{aligned}$$

It completes the proof. ∎

Observe that by monotonicity

$$\lim_{n \to \infty} W_{n,j} = \lim_{n \to \infty} \frac{1 - f_{n,j}\left(0\right)}{e^{S_n - S_j}} = W_j \quad \text{a.s.} \tag{43}$$

and $W_j \overset{d}{=} W$, $j = 1, 2, ...$ where $\mathbf{P}(W \in (0, 1]) = 1$ in view of conditions (9) and Theorem 5 in [6] II.

We can state now the key result:

**Lemma 17** *Assume that Hypotheses A and B are valid and let g be the function satisfying (31). Then,*

$$\lim_{n\to 0}\Bigg| b_n^{-1}\mathbf{E}[Y_j Y_{j,n}\left(1-f_{0,n}(0)\right)e^{-\rho S_n}; X_j \geq \delta an] \\ -\mathbf{E}\left[Y_j e^{-\rho S_{j-1}}\int_{-\infty}^{\infty} Y_{n,j}(v)\left(1-f_{0,j-1}(g_j(e^v W_{n,j}))\right)e^{-\rho v}dv\right]\Bigg| = 0$$

*where* $(W_{n,j}, f_k : k \geq j+1)$, $g_j$ *and* $(S_{j-1}, f_{0,j-1})$ *are independent and*

$$0 < \lim_{n\to\infty}\mathbf{E}\left[e^{-\rho S_{j-1}}\int_{-\infty}^{\infty}\left(1-f_{0,j-1}(g_j(e^v W_{n,j}))\right)e^{-\rho v}dv\right] \\ = \mathbf{E}\left[e^{-\rho S_{j-1}}\int_{-\infty}^{\infty}\left(1-f_{0,j-1}(g_j(e^v W_j))\right)e^{-\rho v}dv\right] < \infty. \quad (44)$$

**Proof.** Introduce the event

$$\mathcal{T}_{N,K,M}(j) = \left\{|S_{j-1}| \leq N,\, |S_n - S_{j-1}| \leq K,\, |X_j - an| \leq M\sqrt{n}\right\}.$$

Recalling that $Y_j$ and $Y_{j,n}$ are bounded, to prove the lemma it is sufficient to study only the quantity

$$\mathbf{E}\left[Y_j Y_{j,n}\left(1-f_{0,n}(0)\right)e^{-\rho S_n}; \mathcal{T}_{N,K,M}(j)\right] \\ = \mathbf{E}[Y_j Y_{n,j}[1-f_{0,j-1}(f_j(f_{n,j}(0)))]e^{-\rho S_j}e^{-\rho S_{n,j}}; \mathcal{T}_{N,K,M}(j)].$$

Moreover, we may assume without loss of generality that $Y_j$ and $Y_{j,n}$ are non-negative. The general case may be considered by writing $Y_j Y_{j,n} = (Y_j Y_{j,n})^+ - (Y_j Y_{j,n})^-$, where $x^+ = \max(x,0)$ and $x^- = -\min(x,0)$.

Clearly,

$$\left\{X_j \geq an - M\sqrt{n}, |S_n - S_{j-1}| \leq K\right\} \subset \left\{S_n - S_j \leq K - an + M\sqrt{n}\right\}.$$

This, in view of the inequality

$$e^{S_{n,j}}W_{n,j} = 1 - f_{n,j}(0) \leq e^{S_{n,j}}$$

and the representation $e^{-x} = 1 - x + o(x)$, $x \to 0$, means that if the event $\mathcal{T}_{N,K,M}(j)$ occurs then, for any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that for all $n \geq n_0$

$$e^{-(1+\varepsilon)(1-f_{n,j}(0))} \leq f_{n,j}(0) \leq e^{-(1-f_{n,j}(0))}.$$

As a result we have

$$\mathbf{E}\left[Y_j Y_{n,j}\left(1-f_{0,j-1}(f_j(e^{-(1-f_{n,j}(0))}))\right)e^{-\rho S_{j-1}}e^{-\rho S_{n,j-1}}; \mathcal{T}_{N,K,M}(j)\right] \\ \leq b_n^{-1}\mathbf{E}\left[Y_j Y_{j,n}\left(1-f_{0,n}(0)\right)e^{-\rho S_n}; \mathcal{T}_{N,K,M}(j)\right] \\ \leq \mathbf{E}\left[Y_j Y_{n,j}\left(1-f_{0,j-1}(f_j(e^{-(1+\varepsilon)(1-f_{n,j}(0))}))\right)e^{-\rho S_{j-1}}e^{-\rho S_{n,j-1}}; \mathcal{T}_{N,K,M}(j)\right].$$

Hence, denoting by $\mathcal{F}_{j-1}$ the $\sigma$-algebra generated by the sequence

$$(f_1, ..., f_{j-1}; S_1, ..., S_{j-1}),$$

we set

$$F_{n,j}(h, K, M; \varepsilon) = \mathbf{E}\left[e^{-\rho(S_n - S_{j-1})} Y_{n,j} \bar{h}\left(f_j\left(\exp\left\{-(1+\varepsilon)\, e^{S_n - S_j} W_{n,j}\right\}\right)\right); B_{j,n}\right],$$

$$O_{n,j}(h, K, M; \varepsilon) = \int_{-K}^{K} e^{-\rho v} dv \mathbf{E}\left[Y_{n,j}(v) \bar{h}\left(g_j\left((1+\varepsilon)\, e^v W_{n,j}\right)\right); \sigma G_{n,j} \in [-M, M]\right]$$

and introduce the random variables

$$\hat{F}_{n,j}(f_{0,j-1}, K, M; \varepsilon) = \mathbf{E}\left[F_{n,j}(f_{0,j-1}, K, M; \varepsilon) | \mathcal{F}_{j-1}\right]$$

and

$$\hat{O}_{n,j}(f_{0,j-1}, K, M; \varepsilon) = \mathbf{E}\left[O_{n,j}(f_{0,j-1}, K, M; \varepsilon) | \mathcal{F}_{j-1}\right].$$

We get from the previous inequalities

$$\begin{aligned} \mathbf{E}&\left[Y_j e^{-\rho S_{j-1}} \hat{F}_{n,j}(f_{0,j-1}, K, M; 0); |S_{j-1}| \leq N\right] \\ &\leq b_n^{-1} \mathbf{E}\left[Y_j Y_{j,n}\left(1 - f_{0,n}(0)\right) e^{-\rho S_n}; \mathcal{T}_{N,K,M}(j)\right] \\ &\leq \mathbf{E}[Y_j e^{-\rho S_{j-1}} \hat{F}_{n,j}(f_{0,j-1}, K, M; \varepsilon); |S_{j-1}| \leq N]. \end{aligned} \tag{45}$$

Moreover the dominated convergence theorem and Lemma 16 give for any fixed $\alpha \in \{0, \varepsilon\}$,

$$\begin{aligned} \limsup_{n\to\infty} \big| b_n^{-1} \mathbf{E}[Y_j e^{-\rho S_{j-1}} \hat{F}_{n,j}(f_{0,j-1}, K, M; \alpha); |S_{j-1}| \leq N] \\ - \mathbf{E}[Y_j e^{-\rho S_{j-1}} \hat{O}_{n,j}(f_{0,j-1}, K, M; \alpha); |S_{j-1}| \leq N] \big| = 0. \end{aligned}$$

Finally, $Y_j$ and $Y_{n,j}(v)$ are bounded (say by 1 for convenience) and we get

$$\begin{aligned} \limsup_{n\to\infty} &\big| \mathbf{E}[Y_j e^{-\rho S_{j-1}} \hat{O}_{n,j}(f_{0,j-1}, K, M; \varepsilon); |S_{j-1}| \leq N] \\ &\quad - \mathbf{E}[Y_j e^{-\rho S_{j-1}} \hat{O}_{n,j}(f_{0,j-1}, K, M; 0); |S_{j-1}| \leq N] \big| \\ &\leq \limsup_{n\to\infty} \mathbf{E}\Big[e^{-\rho S_{j-1}} \int_{-K}^{K} e^{-\rho v} dv \mathbf{E}\big[f_{0,j-1}\big(g_j\left((1+\varepsilon) e^{-v} W_{n,j}\right)\big) \\ &\qquad - f_{0,j-1}\left(g_j\left(e^{-v} W_{n,j}\right)\right)\big]; |S_{j-1}| \leq N\Big] \\ &= \mathbf{E}\Big[e^{-\rho S_{j-1}} \int_{-K}^{K} e^{-\rho v} dv \mathbf{E}\big[f_{0,j-1}\left(g_j\left((1+\varepsilon) e^{-v} W_j\right)\right) - \\ &\qquad f_{0,j-1}\left(g_j\left(e^{-v} W_j\right)\right)\big]; |S_{j-1}| \leq N\Big] \end{aligned}$$

goes to 0 as $\epsilon \to 0$ by monotonicity. We combine the last limits with (45) to get

$$\limsup_{n\to\infty} \big| b_n^{-1} \mathbf{E}\left[ Y_j Y_{n,j} \left(1 - f_{0,n}(0)\right) e^{-\rho S_n}; \mathcal{T}_{N,K,M}(j) \right] \\ - \mathbf{E}[Y_j e^{-\rho S_{j-1}} \hat{O}_{n,j}(f_{0,j-1}, K, M; 0); |S_{j-1}| \le N] \big| = 0. \quad (46)$$

By Corollary 11 and Lemmas 6 (ii), 13, and 14, the fact that $Y_j$ and $Y_{n,j}$ are bounded ensure that

$$\begin{aligned} & \mathbf{E}\left[ Y_j Y_{j,n} \left(1 - f_{0,n}(0)\right) e^{-\rho S_n}; X_j \ge \delta a n \right] \\ = \; & \mathbf{E}[Y_j Y_{j,n} \left(1 - f_{0,n}(0)\right) e^{-\rho S_n}; |S_{j-1}| \le N, X_j \ge \delta a n] + \varepsilon_{N,n} b_n \\ = \; & \mathbf{E}[Y_j Y_{n,j} \left(1 - f_{0,n}(0)\right) e^{-\rho S_n}; |S_{j-1}| \le N, |S_n - S_{j-1}| \le K, X_j \ge \delta a n] + \varepsilon_{N,K,n} b_n \\ = \; & \mathbf{E}[Y_j Y_{n,j} \left(1 - f_{0,n}(0)\right) e^{-\rho S_n}; \mathcal{T}_{N,K,M}(j)] + \varepsilon_{N,K,M,n}(j) b_n, \end{aligned} \quad (47)$$

where

$$\lim_{N\to\infty} \limsup_{K\to\infty} \limsup_{M\to\infty} \limsup_{n\to\infty} |\varepsilon_{N,K,M,n}(j)| = 0. \quad (48)$$

Taking now $Y_j = Y_{n,j} \equiv 1$, adding that $\mathbf{E}\left[ \left(1 - f_{0,n}(0)\right) e^{-\rho S_n} \right] = O\left(b_n\right)$ by Corollary 9 and recalling (46), we deduce, again by monotonicity that

$$\begin{aligned} & \lim_{N\to\infty} \lim_{K\to\infty} \lim_{M\to\infty} \limsup_{n\to\infty} \mathbf{E}[e^{-\rho S_{j-1}} \hat{O}_{n,j}(f_{0,j-1}, K, M; 0); |S_{j-1}| \le N] \\ & \quad = \mathbf{E}\left[ e^{-\rho S_{j-1}} \int_{-\infty}^{\infty} \left(1 - f_{0,j-1}(g_j(e^v W_j))\right) e^{-\rho v} dv \right] \\ & \quad \le \limsup_{n\to\infty} b_n^{-1} \mathbf{E}\left[ \left(1 - f_{0,n}(0)\right) e^{-\rho S_n} \right] \le C < \infty, \end{aligned}$$

proving, in particular, the estimate from above in (44). This, in turn, implies for arbitrary uniformly bounded $Y_j$ and $Y_{n,j}$,

$$\begin{aligned} & \limsup_{n\to\infty} \mathbf{E}[Y_j e^{-\rho S_{j-1}} \hat{O}_{n,j}(f_{0,j-1}, \infty, \infty; 0)] \\ & \qquad \le C \mathbf{E}\left[ e^{-\rho S_{j-1}} \int_{-\infty}^{\infty} \left(1 - f_{0,j-1}(g_j(e^v W_j))\right) e^{-\rho v} dv \right] < \infty \end{aligned}$$

and

$$\limsup_{n\to\infty} \big| b_n^{-1} \mathbf{E}\left[ Y_j Y_{j,n} \left(1 - f_{0,n}(0)\right) e^{-\rho S_n}; X_j \ge \delta a n \right] - \\ - \mathbf{E}[Y_j e^{-\rho S_{j-1}} \hat{O}_{n,j}(f_{0,j-1}, \infty, \infty; 0)] \big| = 0.$$

It yields the first part of the Lemma. We have already checked the finiteness of the limit in (44). Positivity follows from conditions (9), since under these conditions $W > 0$ with probability 1 according to Theorem 5 [6], II. This gives the whole result. ■

# 6 Proof of the Theorems and the Corollary

Now we have an important corollary, which, in fact, proves Theorem 1 with the explicit form of the constant $C_0$ mentioned in the statement of the theorem.

**Proof of Theorem 1.** We assume that Hypotheses A and B are valid. It follows from (37) that for each fixed $j$

$$\mathbf{E}\left[(1-f_{0,n}(0))\exp(-\rho S_n);\mathcal{D}_N(j)\right]=\mathbf{E}[(1-f_{0,n}(0))\,e^{-\rho S_n};X_j\geq\delta an]+\varepsilon_{N,n}b_n.$$

Using this fact and Lemmas 17 and 12 we get

$$\begin{aligned}\lim_{n\to\infty}m^{-n}b_n^{-1}\mathbb{P}\left(Z_n>0\right)&=\lim_{n\to\infty}m^{-n}b_n^{-1}\mathbb{E}\left[(1-f_{0,n}(0))\right]\\&=\lim_{n\to\infty}b_n^{-1}\mathbf{E}\left[(1-f_{0,n}(0))\exp(-\rho S_n)\right]=C_0.\end{aligned}$$

To complete the proof it remains to observe that in view of (17)

$$\begin{aligned}b_n&=\beta\frac{\mathbf{P}\left(X>an\right)}{an}\sim\frac{1}{m}\frac{l_0(an)}{(an)^{\beta+1}}=\frac{1}{m}\frac{l_0(an)}{(an)^{\beta+1}}e^{-\rho an}e^{\rho an}\\&\sim\frac{\rho}{m}e^{\rho an}\int_{an}^{\infty}p_X(x)dx=\frac{\rho}{m}e^{\rho an}\mathbb{P}\left(X>an\right).\end{aligned}$$

Thus,

$$\mathbb{P}\left(Z_n>0\right)\sim C_0m^nb_n\sim C_0\rho m^{n-1}\mathbb{P}\left(X>an\right)e^{an\rho}$$

where, recalling that $g_j, W_j$ and $f_{0,j-1}$ are independent

$$\begin{aligned}C_0&=\sum_{j=1}^{\infty}\mathbf{E}\left[e^{-\rho S_{j-1}}\int_{-\infty}^{\infty}\left(1-f_{0,j-1}(g_j(e^vW_j))\right)e^{-\rho v}dv\right]\qquad(49)\\&=\sum_{j=1}^{\infty}\left(\mathbb{E}\left[e^{\rho X}\right]\right)^{-j+1}\int_{-\infty}^{\infty}\mathbb{E}\left[1-f_{0,j-1}(g_j(e^vW_j))\right]e^{-\rho v}dv.\end{aligned}$$

The proof of the first Theorem is achieved. ■

**Proof of Theorem 2.** Let

$$W_{n,j}(s)=\frac{1-f_{n,j}(s)}{e^{S_n-S_j}},\ s\in[0,1).$$

By monotonicity

$$\lim_{n\to\infty}W_{n,j}(s)=W_j(s)$$

and $W_j(s)\overset{d}{=}W(s)$, $j=1,2,...$ where $\mathbf{P}(W(s)\in(0,1])=1$ thanks to [6] II Theorem 5.

Similarly to Lemma 17 one can show that, as $n \to \infty$

$$\begin{aligned}
&\lim_{n\to\infty} b_n^{-1}\mathbf{E}[(1-f_{0,n}(s))\, e^{-\rho S_n}] \\
&\quad = \lim_{n\to\infty} b_n^{-1}\sum_{j=1}^{\infty}\mathbf{E}[(1-f_{0,n}(s))\, e^{-\rho S_n}; X_j \geq \delta an] \\
&\quad = \mathbf{E}\left[e^{-\rho S_{j-1}}\int_{-\infty}^{\infty}(1-f_{0,j-1}(g(e^v W(s))))\, e^{-\rho v} dv\right] = \Omega_0(s).
\end{aligned}$$

Hence we get

$$\begin{aligned}
\lim_{n\to\infty}\mathbb{E}\left[s^{Z_n}|Z_n>0\right] &= 1-\lim_{n\to\infty}\frac{\mathbf{E}[(1-f_{0,n}(s))\, e^{-\rho S_n}]}{\mathbf{E}[(1-f_{0,n}(0))\, e^{-\rho S_n}]} \\
&= 1-C_0^{-1}\Omega_0(s) =: \Omega(s).
\end{aligned}$$

Theorem 2 is proved. ■

**Proof of Theorem 3.** Coming back to the original probability $\mathbb{P}$, Lemma 17 yields

$$\begin{aligned}
\lim_{n\to\infty}\Big| & b_n^{-1}m^{-n}\mathbb{E}[Y_j Y_{j,n}\mathbb{P}_{\mathfrak{e}}(Z_n>0); X_j\geq \delta an] \\
& -m^{-j-1}\mathbb{E}\left[Y_j\int_{-\infty}^{\infty}Y_{n,j}(v)\,(1-f_{0,j-1}(g_j(e^v W_{n,j})))\, e^{-\rho v} dv\right]\Big| = 0
\end{aligned}$$

Recalling that $\mathbb{P}(Z_n>0)\sim C_0 m^n b_n$ as $n\to\infty$ ensures that

$$\begin{aligned}
\lim_{n\to\infty}\Big| & \mathbb{E}[Y_j Y_{j,n}; X_j\geq \delta an|Z_n>0] \\
& -C_0^{-1}m^{-j-1}\mathbb{E}\left[Y_j\int_{-\infty}^{\infty}Y_{n,j}(v)\,(1-f_{0,j-1}(g_j(e^v W_{n,j})))\, e^{-\rho v} dv\right]\Big| = 0
\end{aligned}$$

We obtain the first part of the Theorem by letting $Y_j = 1$ and $Y_{j,n} = 1$ and using (43), whereas the second part comes by dividing the last displayed formula by $\mathbb{P}(X_j \geq \delta an|Z_n > 0)$. ■

**Proof of the Corollary.** We first check that conditionally on $Z_n > 0$, there is only one big jump. Recalling from Section 5.2 the notation $\mathcal{C}_N = \{-N < S_{\tau_n} \leq S_n \leq N + S_{\tau_n} < N\}$ and the inequality $\mathbf{P}_{\mathfrak{e}}(Z_n > 0)\exp(-\rho S_n) \leq 1$

justified by (40) we have

$$\begin{aligned}
&\mathbb{P}(Z_n > 0, \cup_{i \neq j}^n \{X_i \geq an/2, X_j \geq an/2\}) \\
&\quad = m^n \mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right) \exp(-\rho S_n); \cup_{i \neq j}^n \{X_i \geq an/2, X_j \geq an/2\}\right] \\
&\quad \leq m^n \Bigg( \mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right) \exp(-\rho S_n); \bar{\mathcal{C}}_N\right] \\
&\qquad\qquad + \mathbf{E}\left[L_n \geq -N, S_n \geq N, \cup_{i \neq j}^n \{X_i \geq an/2, X_j \geq an/2\}\right] \Bigg).
\end{aligned}$$

Then Lemma 10 and the limiting relation (29) ensure that

$$\limsup_{n\to\infty} (b_n m^n)^{-1} \mathbb{P}\left(Z_n > 0, \cup_{i \neq j}^n \{X_i \geq an/2, X_j \geq an/2\}\right) = 0.$$

Thus, $\lim_{n\to\infty} \mathbb{P}(\cup_{i \neq j}^n \{X_i \geq an/2, X_j \geq an/2\} | Z_n > 0) = 0$. The first part of the Corollary is then a direct consequence of Theorem 3 (i).

Since $X_j = (S_n - S_{j-1}) - (X_{j+1} + \ldots + X_n)$, the second part is obtained from Theorem 3 (ii) with $F(\cdot) = 1$, $F_{n-j}(v, x_{j+1}, \ldots, x_n) = H((v - x_{j+1} \ldots - x_n - an)/\sqrt{n})$, where $H$ is measurable and bounded. ■

**Acknowledgement.** This work was partially funded by the project MANEGE 'Modèles Aléatoires en Écologie, Génétique et Évolution' 09-BLAN-0215 of ANR (French national research agency), Chair Modelisation Mathematique et Biodiversite VEOLIA-Ecole Polytechnique-MNHN-F.X. and the professorial chair Jean Marjoulet. The second author was also supported by the Program of the Russian Academy of Sciences "Dynamical systems and control theory".

# References

[1] Afanasyev V. I. (1998). Limit theorems for a moderately subcritical branching process in a random environment. *Discrete Math. Appl.,* **8**, pp. 55–62.

[2] Afanasyev V.I., Boeinghoff C., Kersting G., and Vatutin V.A. (2012) Limit theorems for weakly subcritical branching processes in random environment. *J.Theor.Probab.,* **25,** N 3, pp. 703–732.

[3] Afanasyev V.I., Boeinghoff C., Kersting G., and Vatutin V.A. (2013) Conditional limit theorems for intermediately subcritical branching processes in random environment. *Ann. Inst. H. Poincaré Probab. Statist.*, In print, arXiv:1108.2127 [math.PR]

[4] Afanasyev V. I., Geiger J., Kersting G., and Vatutin V. A. (2005). Criticality for branching processes in random environment. *Ann. Probab.* **33**, pp.645–673.

[5] Afanasyev V. I., Geiger J., Kersting G., and Vatutin V. A. (2005). Functional limit theorems for strongly subcritical branching processes in random environment. *Stoch. Proc. Appl.,* **115**, pp.1658–1676.

[6] Athreya K.B., and Karlin S. (1971). On branching processes with random environments: I, II, *Ann. Math. Stat.,* **42**, pp.1499–1520, pp.1843–1858.

[7] Bansaye V., and Vatutin V. (2013). Random walk with heavy tail and negative drift conditionned by its minimum and final values. Avialable via *Arxiv*, http://arxiv.org/abs/1312.3306.

[8] Bingham N.H., Goldie C.M., and Teugels J.L. (1987). *Regular variation.* Cambridge University Press, Cambridge.

[9] Birkner M., Geiger J., and Kersting G. (2005). Branching processes in random environment - a view on critical and subcritical cases. Proceedings of the DFG-Schwerpunktprogramm *Interacting Stochastic Systems of High Complexity*, Springer, Berlin, 265–291.

[10] Borovkov A.A., and Borovkov K.A. (2008). *Asymptotic analysis of random walks. Heavy-tailed distributions.* Encyclopedia of Mathematics and its Applications, 118. Cambridge University Press, Cambridge.

[11] Chover J., Ney P., and Wainger S. (1973). Functions of Probability measures. *J. Analyse Math.,* **26**, pp. 255–302.

[12] Durrett, R. (1980). Conditioned limit theorems for random walks with negative drift. *Z. Wahrsch. Verw. Gebiete* 52, no. **3**, 277-287.

[13] Feller W. (1971). *An Introduction to Probability Theory and Its Applications, Volume II.* John Wiley and Sons, New York.

[14] Geiger J., Kersting G., and Vatutin V.A. (2003). Limit theorems for subcritical branching processes in random environment. *Ann. I.H. Poincaré (B).* **39**, pp. 593–620.

[15] Hirano K. (1998). Determination of the Limiting Coefficient for Exponential Functionals of Random Walks with Positive Drift. *J. Math. Sci. Univ. Tokyo,* **5** , pp. 299–332.

[16] Smith W.L., and Wilkinson W.E. (1969). On branching processes in random environments. *Ann. Math. Stat.,* **40**, pp. 814–827.

[17] Vatutin V., and Zheng X. (2012). Subcritical branching processes in random environment without Cramer condition. *Stochastic Process. Appl.*, **122**, pp. 2594–2609.